A SIMPLIFIED CALCULATION FOR THE FUNDAMENTAL SOLUTION TO THE HEAT EQUATION ON THE HEISENBERG GROUP

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Abstract. Let $\mathcal{L}_\gamma = -1/4\left(\sum_{j=1}^n (X_j^2 + Y_j^2) + i\gamma T\right)$ where $\gamma \in \mathbb{C}$, and $X_j,\ Y_j$ and T are the left invariant vector fields of the Heisenberg group structure for $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. We explicitly compute the Fourier transform (in the spatial variables) of the fundamental solution of the Heat Equation $\partial_s \rho = -\mathcal{L}_\gamma \rho$. As a consequence, we have a simplified computation of the Fourier transform of the fundamental solution of the $\Box_b$-heat equation on the Heisenberg group and an explicit kernel of the heat equation associated to the weighted $\overline{\partial}$-operator in $\mathbb{C}^n$ with weight $\exp(-\tau P(z_1,\dots,z_n))$ where $P(z_1,\dots,z_n) = \frac{1}{2}(|\operatorname{Im} z_1|^2 + \cdots |\operatorname{Im} z_n|^2)$ and $\tau \in \mathbb{R}$.

0. Introduction

The purpose of this note is to present a simplified calculation of the Fourier transform of fundmental solution of the $\Box_b$-heat equation on the Heisenberg group. The Fourier transform of the fundamental solution has been computed by a number of authors [Gav77, Hul76, CT00, Tie06]. We use the approach of [CT00, Tie06] and compute the heat kernel using Hermite functions but differ from the earlier approaches by working on a different, though biholomorphically equivalent, version of the Heisenberg group. The simplification in the computation occurs because the differential operators on this equivalent Heisenberg group take on a simpler form. Moreover, in the proof of Theorem 1.2, we reduce the n-dimensional heat equation to a 1-dimensional heat equation, and this technique would also be useful when analyzing the heat equation on the nonisotropic Heisenberg group (e.g., see [CT00]). We actually use the same version of the Heisenberg group as Hulanicki [Hul76], but he computes the fundamental solution of the heat equation associated to the sub-Laplacian and not the Kohn Laplacian acting on $(0,q)$-forms.

A consequence of our fundamental solution computation is that we can explicitly compute the heat kernel associated to the weighted $\overline{\partial}$-problem in $\mathbb{C}^n$ when the weight is given by $\exp(-\tau P(z_1,\dots,z_n))$ where $\tau \in \mathbb{R}$ and $P(z_1,\dots,z_n) = \frac{1}{2}(|\operatorname{Im} z_1|^2 + \cdots |\operatorname{Im} z_n|^2)$. When $n = 1$ and $p(z_1)$ is subharmonic, nonharmonic polynomial, the weighted $\overline{\partial}$-problem (with weight $\exp(-p(z_1)))$ and explicit construction of Bergman and Szegö kernels and has been studied by a number of authors in different contexts (for example, see [Chr91, Has94, Has95,

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Has98, FS91, Ber92]). In addition, Raich has estimated the heat kernel and its derivatives [Rai06b, Rai06a, Rai07, Rai].

1. The Heisenberg Group and the $\Box_b$-heat equation

Definition 1.1. *The Heisenberg group is the set $\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the following group structure:*

$$g * g' = (x, y, t) * (x', y', t') = (x + x', y + y', t + t' + x \cdot y')$$

where $(x, y, t), (x', y', t') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $\cdot$ denotes the standard dot product in $\mathbb{R}^n$.

The left-invariant vector fields for this group structure are:

$$X_j^g = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t} \text{ and } Y_j^g = \frac{\partial}{\partial y_j}, \quad 1 \le j \le n, \quad \text{and } T^g = \frac{\partial}{\partial t}.$$

The Heisenberg group also can be identified with the following hypersurface in $\mathbb{C}^{n+1}$: $H^n = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} = (1/2)\sum_{j=1}^n (\operatorname{Im} z_j)^2\}$ where we identify $(z_1, \ldots, z_n, t + i(1/2)\sum_{j=1}^n (\operatorname{Im} z_j)^2) \in H^n$ with $(z_1, \ldots, z_n, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ where $z_j = x_j + iy_j \in \mathbb{C}$. With this identification, the left-invariant vector fields of type (0,1) and (1,0), respectively are:

$$\overline{Z}_j^g = (1/2)(X_j + iY_j) = \frac{\partial}{\partial \overline{z}_j} + \frac{y_j}{2}\frac{\partial}{\partial t}, \qquad Z_j^g = (1/2)(X_j - iY_j) = \frac{\partial}{\partial z_j} + \frac{y_j}{2}\frac{\partial}{\partial t}$$

for $g = (x, y, t) \in \mathbb{H}^n$ and $1 \le j \le n$.

The Heat Equation. The Kohn Laplacian $\Box_b$ acting on $(0, q)$-forms on $H^n \approx \mathbb{H}^n$ can be easily described in terms of these left-invariant vector fields. Suppose $f = \sum_{J \in \mathcal{I}_q} f_J d\overline{z}_J$ is a $(0, q)$-form where $\mathcal{I}_q$ is the set of all increasing q-tuples $J = (j_1, \ldots, j_q)$, $1 \le j_k \le n$. Then

$$\Box_b f = \sum_{J \in \mathcal{I}_q} \mathcal{L}_{n-2q} f_J \, d\overline{z}_J$$

where

$$\mathcal{L}_\gamma = -\frac{1}{4}\left(\sum_{j=1}^n (X_j^2 + Y_j^2) + i\gamma T\right). \tag{1}$$

See Stein ([Ste93], XIII §2), for details on computing $\Box_b$. For comparison, the box operator ((or Laplacian) in Hulanicki ([Hul76]) is $-\frac{1}{2}\sum_{j=1}^n (X_j^2 + Y_j^2)$.

The Heat Equation is defined on $(0, q)$-forms ρ on $\mathbb{H}^n$ with coefficient functions that depend on $s \in (0, \infty)$ and $(x, y, t) \in \mathbb{H}^n$. It is

$$\frac{\partial \rho}{\partial s} = -\Box_b \rho$$

(note that here, s is the "time" variable and t is a spatial variable). Since $\Box_b$ acts diagonally, we can restrict ourselves to a fixed component and look for a fundamental solution ρ that satisfies

$$\begin{cases} \dfrac{\partial \rho}{\partial s} = -\mathcal{L}_\gamma \rho \quad \text{for} \quad s > 0,\ (x,y,t) \in \mathbb{H}^n \\ \rho(s=0,x,y,t) = \delta_0(x,y,t) \end{cases} \tag{2}$$

(i.e., the delta function at the origin in the spatial variables).

Fourier Transformed Variables. We will use a Fourier transform in the spatial (x,y,t) variables (i.e. *not* the s-variable): let (α,β,τ) be the transform variables corresponding to (x,y,t), and define:

$$\widehat{f}(\alpha,\beta,\tau) = \int_{\mathbb{H}^n} f(x,y,t)\, e^{-i(\alpha\cdot x+\beta\cdot y+\tau t)}\, dx\, dy\, dt.$$

Our main result is the following:

Theorem 1.2. *For any $\gamma \in \mathbb{C}$, the spatial Fourier transform of the fundamental solution to the heat equation (2) is given by*

$$\hat{\rho}^\gamma(s,\alpha,\beta,\tau) = \frac{e^{-\gamma s\tau/4}}{(\cosh(s\tau/2))^{n/2}} e^{-A(|\alpha|^2+|\beta|^2)/2+iB\alpha\cdot\beta}. \tag{3}$$

where

$$A = \frac{\sinh(s\tau/2)}{\tau\cosh(s\tau/2)}, \qquad B = \frac{2\sinh^2(s\tau/4)}{\tau\cosh(s\tau/2)}.$$

Note that γ may be any complex number, but $\gamma = n-2q$ is the value where $\mathcal{L}_\gamma$ corresponds to $\Box_b$ on $(0,q)$-forms.

We also seek the fundamental solution to the heat equation associated to the weighted $\overline{\partial}$ operator in (s,x,y)-space. Given a function f on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, let

$$\tilde{f}_\tau(x,y) = \int_{\mathbb{R}} e^{-i\tau t} f(x,y,t)\, dt$$

be the partial Fourier transform in t. Define

$$\overline{L}_j = \frac{\partial}{\partial \overline{z}_j} + \frac{i}{2} y_j\tau = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j} + iy\tau\Big), \quad L_j = \frac{\partial}{\partial z_j} + \frac{i}{2} y_j\tau = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j} + iy\tau\Big).$$

Note that these operators are just the Fourier transform of $\overline{Z}_j$ and Z_j in the t-direction. If $\triangle_{x,y}$ is the Laplacian in both the x and y variables, the partial t-Fourier transform of $\mathcal{L}_\gamma$ is

$$\tilde{\mathcal{L}_\gamma} = -\frac{1}{4}\big(\triangle_{x,y} + 2i\tau\, y\cdot\nabla_x - (\tau^2 y\cdot y + \gamma\tau)\big).$$

The operator $\tilde{\mathcal{L}_\gamma}$ acts on functions, but it can be extended to $(0,q)$-forms by acting on each component function of the form. If $\gamma = n-2q$, then $\tilde{\mathcal{L}_\gamma}$ is the higher dimensional analog of the $\Box_{\tau p}$-operator from [Rai06a, Rai07, Rai] associated to the weighted $\overline{\partial}$ operator in $\mathbb{C}^n$ with

weight $\exp(-\tau P(z_1,\dots,z_n))$ where $P(z_1,\dots,z_n)=\frac{1}{2}(|\operatorname{Im} z_1|^2+\cdots|\operatorname{Im} z_n|^2)$ and $\tau\in\mathbb{R}$. As a corollary to our main theorem, we compute the fundamental solution to the heat operator associated to this weighted $\overline{\partial}$.

Corollary 1.3. *For any $\gamma\in\mathbb{C}$, $\tau\in\mathbb{R}$, the function*

$$\tilde{\rho}^{\gamma}_{\tau}(s,x,y)=\frac{e^{-\gamma s\tau/4}}{(2\pi)^n(\cosh(s\tau/2))^{n/2}(A^2+B^2)^{n/2}}e^{-\frac{A}{2(A^2+B^2)}(|x|^2+|y|^2)-i\frac{B}{A^2+B^2}x\cdot y}.$$

is the fundamental solution to the weighted $\overline{\partial}$ heat equation: $(\frac{\partial}{\partial s}+\tilde{\mathcal{L}}_{\gamma})\tilde{\rho}^{\gamma}_{\tau}(s,x,y)=0$ with $\tilde{\rho}^{\gamma}_{\tau}(s=0,x,y)=\delta_{(0,0)}(x,y)$.

Finally, we use $\tilde{\rho}^{\gamma}_{\tau}$ to derive the heat kernel, as studied in [Rai06a, Rai07, Rai, NS01].

Corollary 1.4. *For any $\gamma\in\mathbb{C}$, $\tau\in\mathbb{R}$, let*

$$H^{\gamma}_{\tau}(s,x',y',x,y)=\frac{\tau^n e^{-\gamma s\tau/4}}{(4\pi)^n\sinh^n(s\tau/4)}e^{-\frac{\tau}{4}\coth(s\tau/4)\left(|x-x'|^2+|y-y'|^2\right)-i\frac{\tau}{2}(x-x')\cdot(y+y')}.$$

Then H^{γ}_{τ} is the heat kernel which satisfies the following property: if $f\in L^2(\mathbb{C})$, then

$$H^{\gamma}_{\tau}[f](s,x,y)=\int_{\mathbb{R}^n\times\mathbb{R}^n}H^{\gamma}_{\tau}(s,x,y,x',y')f(x',y')\,dx'dy'$$

is a solution to the following initial value problem for the heat equation:

$$\begin{cases}\Big(\dfrac{\partial}{\partial s}+\tilde{\mathcal{L}}_{\gamma}\Big)H^{\gamma}_{\tau}[f]=0\\ H^{\gamma}_{\tau}[f](s=0,x,y)=f(x,y).\end{cases}\tag{4}$$

Note that H^{γ}_{τ} is conjugate symmetric in $z=x+iy$ and $z'=x'+iy'$ (i.e. switching z with z' results in a conjugate).

2. Proof of Theorem 1.2

It is easy to verify the following calculations. Recall that $\widehat{\ }$ refers to spatial Fourier transform.

$$\begin{aligned}\widehat{X_j^2 f}(\alpha,\beta,\tau) &= (-\alpha_j^2-2i\alpha_j\tau\frac{\partial}{\partial\beta_j}+\tau^2\frac{\partial^2}{\partial\beta_j^2})\widehat{f}\\ \widehat{Y_j^2 f}(\alpha,\beta,\tau) &= -\beta_j^2\widehat{f}\\ \widehat{Tf}(\alpha,\beta,\tau) &= i\tau\widehat{f}.\end{aligned}$$

We first reduce the problem down to dimension one. Define $\hat{\rho}^{\gamma,1}$ by the same formula as given in (3), but for dimension one (i.e. $n=1$ and $\alpha,\ \beta\in\mathbb{R}$). From (3), note that

$$\hat{\rho}^{\gamma}(s,\alpha,\beta,\tau)=\prod_{j=1}^{n}\hat{\rho}^{\gamma/n,1}(s,\alpha_j,\beta_j,\tau),\quad \alpha=(\alpha_1,\dots,\alpha_n),\ \beta=(\beta_1,\dots,\beta_n)\in\mathbb{R}^n\tag{5}$$

(note the γ on the left and the γ/n on the right). Once we show that $\rho^{\gamma,1}$ satisfies the transformed heat-equation in dimension one, i.e.,

$$\left(\frac{\partial}{\partial s} - (1/4)(\widehat{X^2} + \widehat{Y^2} + i\gamma\widehat{T})\right)\{\widehat{\rho}^{\gamma,1}(s,\cdot,\cdot)\} = 0 \tag{6}$$

with initial condition $\hat{\rho}^{\gamma,1} = 1$ (the Fourier transform of the delta function), then by using (5), it is an easy exercise to show that $\widehat{\rho}^{\gamma}$ in dimension n satisfies Theorem 1.2.

From now on, we assume the dimension n is one and so x, y, α and β are all real variables. Also, γ will be suppressed as a superscript. Define

$$u(s,\alpha,\beta,\tau) = \widehat{\rho}(s,\alpha,\beta,\tau)e^{-i\frac{\alpha\beta}{\tau}}. \tag{7}$$

Then, the following equations are easily verified

$$u(s=0,\alpha,\beta,\tau) \;=\; e^{-i\frac{\alpha\beta}{\tau}} \tag{8}$$

$$\frac{\partial u}{\partial s} \;=\; (1/4)(\tau^2\frac{\partial^2}{\partial\beta^2} - \beta^2 - \gamma\tau)u. \tag{9}$$

The first equation follows from the fact that the Fourier transform of the delta function is the constant one. The second equation follows from the heat equation for $\widehat{\rho}$ (from (6)) and the above formulas for the transformed differential operators $\widehat{X}, \widehat{Y}$ and $\widehat{T}$. We will refer to the above differential equation as the *transformed Heat equation.*

Solution of Heat Equation Using Hermite Special Functions. For $m = 0, 1, 2 \ldots$ and $x \in \mathbb{R}$, let

$$\psi_m(x) \;=\; \frac{(-1)^m}{\sqrt{2^m m!\sqrt{\pi}}}e^{x^2/2}\frac{d^m}{dx^m}\{e^{-x^2}\}.$$

For $\tau \in \mathbb{R}$, let

$$\Psi_m^\tau(x) = |\tau|^{-1/4}\psi_m(x/\sqrt{|\tau|}).$$

It is a fact that ψ_m and hence Ψ_m^τ form an orthonormal system for $L^2(\mathbb{R})$ (see [Tha93], pg.1-7). It is also a fact (again see [Tha93], (1.1.28)) that

$$\psi_m''(x) = x^2\psi_m(x) - (2m+1)\psi_m(x).$$

We first assume that $\tau > 0$ and later indicate the minor changes needed in the case that $\tau \leq 0$. Replacing x by $\beta/\sqrt{\tau}$ in the previous equation yields:

$$(\tau^2\frac{\partial^2}{\partial\beta^2} - \beta^2 - \gamma\tau)\{\Psi_m^\tau\} = -(2m+1+\gamma)\tau\Psi_m^\tau(\beta). \tag{10}$$

In other words, Ψ_m^τ is an eigenfunction of the differential operator on the right side of (9) with eigenvalue $-(2m+1+\gamma)\tau$.

Since $\{\Psi_m^\tau\}$ are an orthonormal basis for $L^2(\mathbb{R})$, u can be expressed as

$$u(s,\alpha,\beta,\tau)=\sum_{m=0}^{\infty}a_m(\alpha,\tau)e^{-\frac{1}{4}(2m+1+\gamma)s\tau}\Psi_m^\tau(\beta)$$

where $a_m(\alpha,\tau)$ will be determined later. Differentiating this with respect to s and using (10) gives

$$\begin{aligned}\frac{\partial}{\partial s}u(s,\alpha,\beta,\tau)&=\sum_{m=0}^{\infty}a_m(\alpha,\tau)e^{-\frac{1}{4}(2m+1+\gamma)s\tau}(-\frac{1}{4}(2m+1+\gamma))\tau\Psi_m^\tau(\beta)\\&=\frac{1}{4}\left(\tau^2\frac{\partial^2}{\partial\beta^2}-\beta^2-\gamma\tau\right)\{u(t,\alpha,\beta,\tau)\}.\end{aligned}$$

So, u satisfies the transformed Heat equation (9). To satisfy the initial condition (8), we must have

$$e^{-i\alpha\beta/\tau}=u(s=0,\alpha,\beta,\tau)=\sum_{m=0}^{\infty}a_m(\alpha,\tau)\Psi_m^\tau(\beta).$$

Using the fact that the $\Psi_m^\tau(\beta)$ is an orthonormal system, we have

$$a_m(\alpha,\tau)=\int_{\mathbb{R}}e^{-i\alpha\beta/\tau}\Psi_m^\tau(\beta)\,d\beta=\tau^{1/4}\int_{\mathbb{R}}e^{-i\frac{\alpha}{\sqrt{\tau}}\beta}\psi_m(\beta)\,d\beta.$$

The integral on the right is just the Fourier transform of ψ_m at the point $\alpha/\sqrt{\tau}$. From Thangavelu ([Tha93], Lemma 1.1.3), the Fourier transform of ψ_m equals ψ_m up to a constant factor of $(-i)^m\sqrt{2\pi}$. Therefore,

$$a_m(\alpha,\tau)=(-i)^m(2\pi)^{1/2}\tau^{1/4}\psi_m(\alpha/\sqrt{\tau}).$$

Substituting this value of a_m into the expression for u and rearranging gives:

$$u(s,\alpha,\beta,\tau)=(2\pi)^{1/2}e^{-\frac{1}{4}(1+\gamma)s\tau}\sum_{m=0}^{\infty}(-i)^m\psi_m(\frac{\alpha}{\sqrt{\tau}})\psi_m(\frac{\beta}{\sqrt{\tau}})e^{-\frac{1}{2}ms\tau}.$$

Now solving for $\hat{\rho}$ (see equation (7)) yields

$$\hat{\rho}(s,\alpha,\beta,\tau)=e^{i\alpha\beta/\tau}u(s,\alpha,\beta,\tau)=(2\pi)^{1/2}e^{-\frac{1}{4}(1+\gamma)s\tau}\sum_{m=0}^{\infty}(-i)^m\psi_m(\tfrac{\alpha}{\sqrt{\tau}})\psi_m(\tfrac{\beta}{\sqrt{\tau}})e^{-\frac{1}{2}ms\tau}e^{i\alpha\beta/\tau}.$$

Now let $S=e^{-s\tau/2},\ x=\alpha/\sqrt{\tau},\ y=\beta/\sqrt{\tau}$. Since $|iS|<1$, we obtain (see [Tha93], (1.1.36))

$$\begin{aligned}\hat{\rho}(s,\alpha,\beta,\tau)&=(2\pi)^{1/2}S^{\frac{1}{2}(1+\gamma)}\left(\sum_{m=0}^{\infty}(-iS)^m\psi_m(x)\psi_m(y)\right)e^{ixy}\\&=\frac{\sqrt{2}S^{\frac{1}{2}(1+\gamma)}}{(1+S^2)^{1/2}}e^{-\frac{1}{2}\frac{1-S^2}{1+S^2}(x^2+y^2)}e^{ixy(\frac{-2S}{1+S^2}+1)}.\end{aligned}$$

Now substituting in for S, x and y, a short calculation finishes the proof for $\tau>0$. Note that $\hat{\rho}(s=0,\alpha,\beta,\tau)=1$ (the Fourier transform of the delta function at the origin).

When $\tau=0$, the solution in (3) becomes $\widehat{\rho}(s,\alpha,\beta)=e^{-s(\alpha^2+\beta^2)/4}$ which is easily shown to satisfy (6).

If $\tau < 0$, then τ is replaced by $|\tau|$ on the right side of (10), which slightly changes the subsequent calculations. However the formula for the solution given Theorem 1.2 remains valid for $\tau < 0$.

3. Proof of the Corollaries

Proof. (Corollary 1.3). Again, we assume the dimension is $n = 1$. The fundamental solution to this heat operator must satisfy

$$\frac{\partial}{\partial s}\tilde{\rho}_\tau(s,x,y) + \tilde{\mathcal{L}}_\gamma \tilde{\rho}_\tau = 0$$

with the initial condition $\tilde{\rho}_\tau(s=0,x,y) = \delta_0(x,y)$. Now since $\hat{\rho}$ is the Fourier transform of the fundamental solution to the original Heat operator, clearly $\tilde{\rho}_\tau$ can be obtained by taking the inverse Fourier transform of $\hat{\rho}$ in the α, β variables. This is a standard calculation involving Gaussian integrals and will be left to the reader. □

Proof. (Corollary 1.4). If L_j and $\overline{L}_j$, $1 \le j \le n$, had constant coefficients then the heat kernel would just be $\tilde{\rho}_\tau(s, x-x', y-y')$ – an ordinary convolution. However, we must multiply by a "twist" factor $e^{-i\tau(x-x')\cdot y'}$ to account for the fact that L_j and $\overline{L_j}$ have variable coefficients. Let

$$H_\tau(s,x,y,x',y',\tau) = \tilde{\rho}_\tau(s,x-x',y-y')e^{-i\tau(x-x')\cdot y'}. \tag{11}$$

Note that $H_\tau(f)$ satisfies the initial condition given in (4) in view of the initial condition satisfied by $\tilde{\rho}_\tau$ and noting that the twist term is 1 at $x' = x$. Showing that H_τ satisfies the heat equation in the s, x, y variables is a short calculation that uses the equation

$$\left(\frac{\partial}{\partial s} - \frac{1}{4}\Big(\triangle_{x,y} + 2i\tau(y-y')\cdot\nabla_x - (\tau^2(y-y')\cdot(y-y') + \gamma\tau)\Big)\right)\{\tilde{\rho}_\tau(s,x-x',y-y')\} = 0.$$

which is just the equation $(\frac{\partial}{\partial s} + \tilde{\mathcal{L}}_\gamma)\tilde{\rho}_\tau = 0$ at the point $(s, x-x', y-y')$.

Simplification of the Formula for H_τ. Note that the coefficient of the imaginary part of the exponent of $\tilde{\rho}_\tau$ is

$$\frac{-B}{A^2+B^2} \quad \text{where} \quad A = \frac{\sinh(s\tau/2)}{\tau\cosh(s\tau/2)}, \quad B = \frac{2\sinh^2(s\tau/4)}{\tau\cosh(s\tau/2)}.$$

An easy calculation with cosh and sinh identities shows that

$$\frac{B}{A^2+B^2} = \frac{\tau}{2} \quad \text{and} \quad \frac{A}{B} = \frac{\cosh(s\tau/4)}{\sinh(s\tau/4)}.$$

Consequently, the fundamental solution H_τ, from (11) and Corollary 1.3, can be rewritten

$$H_\tau(s,x',y',x,y) = \frac{\tau^n e^{-\gamma s\tau/4}}{(4\pi)^n\sinh^n(s\tau/4)} e^{-\frac{\tau}{4}\coth(s\tau/4)\left(|x-x'|^2+|y-y'|^2\right) - i\frac{\tau}{2}(x-x')\cdot(y+y')}.$$

□

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